THE ANALOGUE OF EBERHARD'S THEOREM FOR 4-VALENT GRAPHS ON THE TORUS*

BY

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ABSTRACT

It is shown that every sequence $\{p_k \mid 3 \leqq k \neq 4\}$ (with one exception) that satisfies the right equation is realizable by a 4-valent graph on the torus.

Let a cell complex C decompose an orientable 2-manifold T_g of genus g, having p_k k-gons for all $k \geqq 3$, and such that the 1-skeleton of C ("the graph of C") is an m-valent graph, $m = 3,4$. It follows from the well-known Euler's equation $V - E + F = 2(1-g)$ that

$$(1) \qquad \sum_{k \geqq 3} (6 - k)p_k = 12(1 - g) \qquad \text{for } m = 3$$

and

$$(2) \qquad \sum_{k \geqq 3} (4 - k)p_k = 8(1 - g) \qquad \text{for } m = 4.$$

A sequence $\{p_k | 3 \leqq k \neq 6\}$ ($\{p_k | 3 \leqq k \neq 4\}$) of non negative integers is said to be 3-*realizable on* T_g (4-*realizable on* T_g) if there is a value for p_6 (p_4, resp.) and a cell decomposition C of T_g, such that C has p_k k-gons for all $k \geqq 3$ and and the graph of C is 3-valent (4-valent, resp.); in addition, in case $g = 0$ T_0 is supposed to be a convex polytope and C its boundary cell complex (for definitions, see [3]); if $g \geqq 1$ and the graph of C is 4-valent, then every 2-face of C is supposed to be a planar convex polygon (this is impossible in cases where $g \geqq 1$ and the graph of C is 3-valent, see [3], p. 206).

We are ready to state

EBERHARD'S THEOREM. *Every sequence* $\{p_k | 3 \leqq k \neq 6\}$ *of non negative integers satisfying* (1) *with* $g = 0$ *is* 3-*realizable on* T (*see* [2], *and also* [3], p. 256).

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GRÜNBAUM'S THEOREMS

(A) *Every sequence* $\{p_k | 3 \leqq k \neq 6,\ p_3 = p_4 = 0\}$ *of non negative integers satisfying* (1) *with* $g = 0$ *is* 3*-realizable on* T_0 *with a preassigned value for* p_6, *provided* $p_6 \geqq 8$ ([4], Th. 1).

(B) *Every sequence* $\{p_k | 3 \leqq k \neq 4\}$ *of non negative integers satisfying* (2) *with* $g = 0$ *is* 4*-realizable on* T_0 ([5], *see also* [3], p. 254).

(C) *Every* sequence* $\{p_k | 3 \leqq k \neq 6,\ p_3 = p_4 = 0\}$ *of non negative integers satisfying* (1) *with* $g = 1$ *is* 3*-realizable on* T_1 *with a preassigned value for* p_6, *provided* $p_6 \geqq 12$ ([4], Th. 3).

The purpose of this paper is to prove the following (see [3], p. 269, problem #7)

MAIN THEOREM. *Every sequence* $\{p_k | 3 \leqq k \neq 4\}$ *of non negative integers satisfying* (2) *with* $g = 1$ *is* 4*-realizable on* T_1, *except for the sequence*

$$\{p_3 = p_5 = 1,\ p_k = 0 \text{ for all } k \geqq 6\}.$$

The exceptional case in the Main Theorem is not 4-realizable on T_1, due to D. Barnette [1].

COROLLARY. *Every sequence* $\{p_k | 3 \leqq k \neq 4\}$ *of non negative integers satisfying* (2) *with* $g \geqq 1$ *and one of the following conditions is* 4*-realizable on* T_g:

(A.1) $p_3 \neq 1$ *and* $p_5 \geqq 8(g-1)$,

or (A.2) $p_3 \neq 1$, $p_5 \geqq 4(g-1)$ *and* $p_6 \geqq 2(g-1)$,

or (A.3) $p_3 \neq 1$, $p_5 \geqq 2(g-1)$ *and* $p_7 \geqq 2(g-1)$,

or (A.4) $p_3 \neq 1$, *and* $p_6 \geqq 4(g-1)$,

or (A.5) $p_3 \neq 1$ *and* $p_8 \geqq 2(g-1)$.

Our proof leans heavily on B. Grünbaum's [5] (see also [3], p. 254); the main tool here is

STEINITZ'S THEOREM [7]: *A graph G is the graph of a* 3*-polytope if and only if G is planar and* 3*-connected.*

We need the following two lemmas.

LEMMA 1. *If R is a convex m-gon in the plane* E^2 *and u is a direction in* E^2, *then there exists a cell complex C decomposing R such that*

(i) *C has exactly one regular m-gon* R_1 *and the rest of its* 2*-cells are* (*convex*) *quadrangles,*

(ii) *one edge of* R_1 *is parallel to u*,

(iii) *the edges of R are edges of C*,

* The proof of Th. 3 in [4] does not cover the case $\{p_5 = p_7 = 1,\ p_k = 0$ for all other $k \neq 6\}$. This case is still open (see also [6]), and it seems to be non 3-realizable on T_1.

and (iv) *all vertices of C are* 4-*valent, except for the verticles of both R and* R_1 *which are* 3-*valent.*

The proof of Lemma 1 is an elementary one, hence it is omitted.

LEMMA 2. *For every sequence* $\{p_k | 3 \leqq k \neq 4\}$ *of non negative integers satisfying equation* (2) *with* $g = 1$, *Except for the sequence* $\{p_3 = p_5 = 1,$ $p_k = 0$ for all $k \geqq 6\}$, *there exists a* 3-*connected planar graph G having the following properties*;

(i) *for some* $m \geqq 3$, *G has* p_k *k-gons for all* $3 \leqq k \neq 4, m$, *and it has* $p_m + 2$ *m-gons*,

(ii) *two of the m-gons R and S are disjoint, and*

(iii) *the vertices of G are* 4-*valent, except for the vertices of R and S which are* 3-*valent.*

PROOF OF LEMMA 2. We will construct a graph G for every sequence $\{p_k | 3 \leqq k \neq 4\}$ mentioned in Lemma 2, using the following two definitions:

A *k-block*, $k \geqq 5$, is a decomposition of a square into one k-gon, $k-4$ triangles and some quadrangles, as shown in Fig. 1.

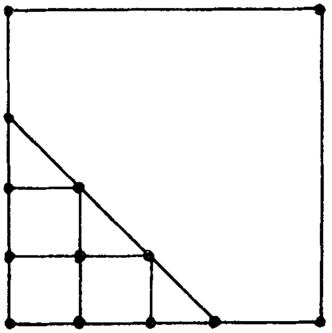

Fig. 1

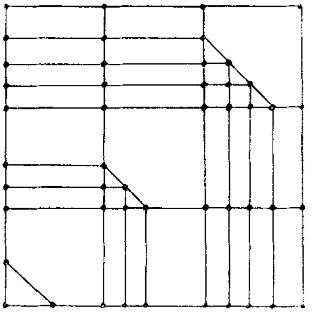

Fig. 2

A *base* is a collection of some k-blocks put together along a diagonal of a bigger square with few additional segments, as shown in Fig. 2.

Case 1. $p_k \neq 0$ for some $k \geqq 6$.

Let $p_{k_0} \neq 0$ for $k_0 \geqq 6$. We construct a base haerving one $(k_0 - 2)$-block F in its upper right corner, and p_k k-blocks for all $5 \leqq k \neq k_0$, and $p_{k_0} - 1$ k_0-blocks; convert the (k_0-2)-gon in F into a k_0-gon by adding two 2-valent vertices A and B, and "close" the base to a graph G, as shown Fig. 3.

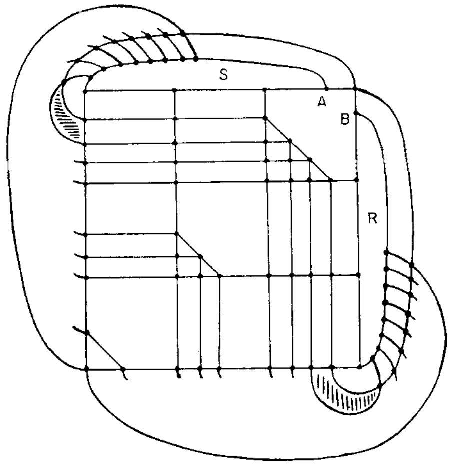

Fig. 3

The two m-gons R and S of G are described in Fig 3: G has, in addition, p_k k-gons for all $k \geqq 5$. As for the number of the triangles of G, there are $k-4$ in each k-block, except for the $k_0 - 6$ triangles in F, and two more triangles of G are outside the base (in shade); therefore G has $\Sigma_{k \geqq 5}\,(k-4)p_k$ triangles as it should have according to equation (2) with $g = 1$.

Case 2. $p_k = 0$ for all $k \geqq 6$.

It therefore follows from Eq. (2) with $g = 1$ that $p_3 = p_5 = n$. We divide this case to two sub-cases.

Sub-case 2.1. n is even.

If $n = 0$, the graph of a prism is as required.

If $n \geqq 2$, we arranged $n/2$ pairs of 5-blocks to a base, each pair being situated opposingly, and we "close" the base to a graph G, as shown in Fig. 4.

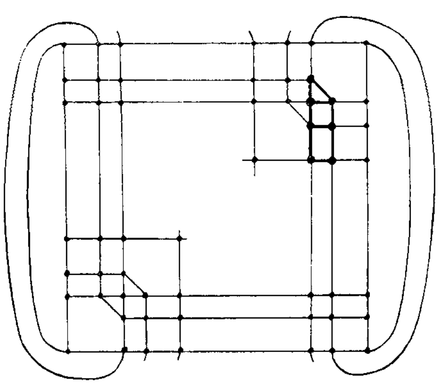

Fig. 4

Sub-case 2.2. n is odd.

Since the case $\{p_3 = p_5 = 1,\ p_k = 0 \text{ for all } k \geqq 6\}$ is excluded, $n \geqq 3$. The graph G in this case is obtained from the corresponding graph in Sub-case 2.1 applied for $n-1$, by replacing the emphasized sub-graph (Fig. 5.1), consisting of one triangle and two quadrangles, with a graph (Fig. 5.2) consisting of two triangles, one pentagon and a quadrangle.

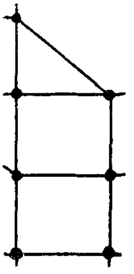

Fig. 5.1

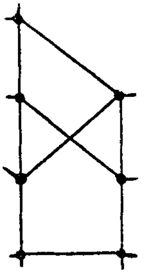

Fig. 5.2

This completes the proof of Lemma 2.

Proof of Main Theorem. Let $\{p_k | 3 \leqq k \neq 4\}$ be a sequence of non negative integers, different from $\{p_3 = p_5 = 1, p_k = 0 \text{ for all } k \geqq 6\}$, satisfying equation (2) with $g = 1$. The graph G, given by Lemma 2, is planar and 3-connected, hence by Steinitz's Theorem there exists a 3-polytope P in E^3, having a graph isomorphic to G. Let R and S be the two m-gons on the boundary of P that correspond to R and S of G.

Applying a projective transformation (that takes a plane M to the plane at infinity, where $M \cap P = \emptyset$ and M contains the intersection of the two planes in which R and S lie) if needed, we may assume as we do that R and S lie in parallel planes.

We apply Lemma 1 to R and S, using a common direction u for both, so that the boundary of P is sub-divided, yielding two regular m-gons R' and S' and some quadrangles.

The promised torus T_1 is given by

$$T_1 = \text{closure } [P - \text{convex } (R' \cup S')],$$

i.e. T_1 is obtained from P by removing the interior of the m-sided prism convex $(R' \cup S')$ together with the interior of R' and S'.

The two m-gons of P were used to get the torus T_1, hence the cell decomposition of T_1 that was obtained from the boundary cell decomposition of P has p_k k-gons, for all $3 \leqq k \neq 4$, and its graph is 4-valent.

This completes the proof of the Main Theorem.

REMARK. E. Jucovič independently proved [6] our Main Theorem, except that it seems to the author of this paper that some quadrangles in Jucovič's construction (of adding a handle to the polytope P, made out of prisms) are not planar and convex.

PROOF OF THE COROLLARY. Let $\{p_k | 3 \leqq k \neq 4\}$ be a sequence of non negative integers satisfying Eq. (2) with some $g \geqq 1$ and condition (A.1).

Define a new sequence $\{q_k | 3 \leqq k \neq 4\}$ by $q_5 = p_5 - 8(g-1)$ and $q_k = p_k$ for all $3 \leqq k \neq 4,5$.

$\{q_k | 3 \leqq k \neq 4\}$ satisfies Eq. (2) with $g = 1$, since

$$\sum_{k \geqq 3} (4-k)q_k = \sum_{k \geqq 3} (4-k)p_k - (4-5)\, 8(g-1) = 8(1-g) - 8(1-g) = 0.$$

As in the proof of the Main Theorem, there exists a 3-polytope P, having two m-gons R and S on parallel planes, and in addition it has q_k k-gons for all $3 \leqq k \neq 4$, while the graph of P is 4-valent, except for the vertices of R and S which are 3-valent. We first subdivide R (S) into $4(g-1)$ pentagons, some quadrangles and g 2-cells $R_1, \dots, R_g$ ($S_1 \dots, S_g$, respectively), such that both R_j and S_j are k_j-gons, for all $1 \leqq j \leqq g$ (see Fig. 6.1). We apply Lemma 1 to every couple R_j and S_j, $1 \leqq j \leqq g$, using a common direction, and take the subdivisions to yield very small R'_j and S'_j, such that convex $(R'_i \cup S'_i) \cap$ convex $(R'_j \cup S'_j) = \varnothing$, for all $i \neq j$, $1 \leqq i, j \leqq g$. The required torus T_g is obtained from P by removing g disjoint prisms:

$$T_g = \text{closure } [P - \bigcup_{i=1}^{g} \text{convex } (R'_i \cup S'_i)].$$

If the given sequence $\{p_k | 3 \leqq k \neq 4\}$ satisfies condition (A.i), a similar proof works, using the sub-division of the corresponding two m-gons R and S as described in Fig. (6.i), $1 \leqq i \leqq 5$.

This completes the proof of the Corollary.

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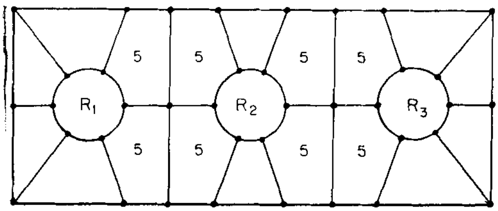

Fig. 6.1

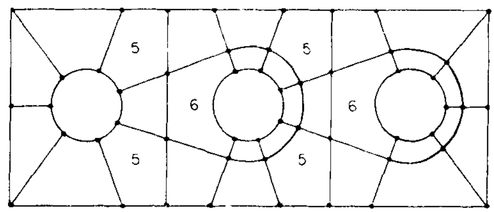

Fig. 6.2

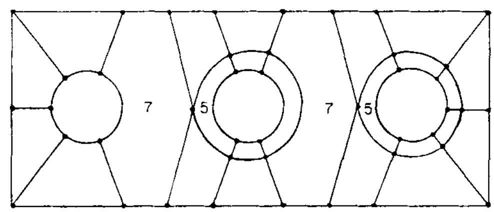

Fig. 6.3

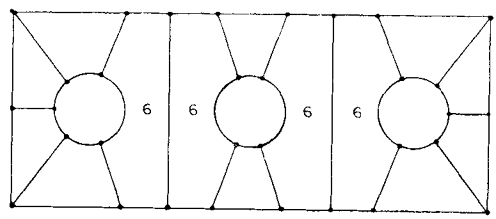

Fig. 6.4

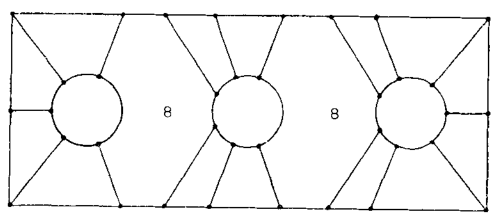

Fig. 6.5

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